Journal of Statistical Physics, Vol. 126, No. 1, January 2007 (© 2006)
DOI: 10.1007/s10955-006-9250-4

Nonequilibrium Gas and Generalized Billiards

Mikhail V. Deryabin[1] and Lev D. Pustyl'nikov[2]

Received April 6, 2006; accepted October 26, 2006
Published Online: December 22, 2006

Generalized billiards describe nonequilibrium gas, consisting of finitely many particles, that move in a container, whose walls heat up or cool down. Generalized billiards can be considered both in the framework of the Newtonian mechanics and of the relativity theory. In the Newtonian case, a generalized billiard may possess an invariant measure; the Gibbs entropy with respect to this measure is constant. On the contrary, generalized relativistic billiards are always dissipative, and the Gibbs entropy with respect to the same measure grows under some natural conditions.

In this article, we find the necessary and sufficient conditions for a generalized Newtonian billiard to possess a smooth invariant measure, which is independent of the boundary action: the corresponding classical billiard should have an additional first integral of special type. In particular, the generalized Sinai billiards do not possess a smooth invariant measure. We then consider generalized billiards inside a ball, which is one of the main examples of the Newtonian generalized billiards which does have an invariant measure. We construct explicitly the invariant measure, and find the conditions for the Gibbs entropy growth for the corresponding relativistic billiard both for monotone and periodic action of the boundary.

KEY WORDS: nonequilibrium gas, The Gibbs entropy, invariant measure, generalized billiards

1. INTRODUCTION

The description of gas as a system of elastic balls moving inside a container goes back to Boltzmann and Poincaré.[1,2] For an ideal gas, the balls, representing its molecules, are replaced by point masses (particles). The probability of collision

[1] The Mads Clausen Institute, University of Southern Denmark, Grundtvigs Allé 150, DK6400 Sønderborg, Denmark; e-mail: mikhail@mci.sdu.dk
[2] Keldysh Institute of Applied Mathematics of RAS, Miusskaja sq. 4, 125047, Moscow, Russia, and University of Bielefeld, BiBoS, Postfach100131, D-33501, Bielefeld, Germany; e-mail: pustylni@Physik.Uni-Bielefeld.De

of two particles is zero, so the particles move independently from each other. Behaviour of such a system is described by billiards that were introduced by Birkhoff:[3] a particle moves linearly and uniformly inside a closed domain Π with a piece-wise smooth boundary $\Gamma = \partial\Pi$, and bounces off the boundary Γ, such that the normal component of its velocity changes the sign, while the tangential component remains the same. The particle energy is a first integral, thus this classical billiard describes equilibrium gases.

In this paper we consider generalized billiards, which is a model of a nonequilibrium gas, introduced in Ref. 4. The essence of the generalization is in the collision law. Let a function $f(\gamma, t)$ be given on the direct product $\Gamma \times \mathbb{R}^1$ (where $\mathbb{R}^1$ is the real line, $\gamma \in \Gamma$ is a point of the boundary and $t \in \mathbb{R}^1$ is time). Suppose that the trajectory of the particle, which moves with the velocity v, intersects Γ at the point $\gamma \in \Gamma$ at time t^*. Then at time t^* the particle acquires the velocity v^*, as if it underwent an elastic push from the infinitely-heavy plane Γ^*, which is tangent to Γ at the point γ, and at time t^* moves along the normal to Γ at γ with the velocity $\frac{\partial f}{\partial t}(\gamma, t^*)$. Here we take the positive direction of motion of the plane Γ^* to be towards the *interior* of the domain Π. We emphasize that the position of the boundary itself is fixed, while its action upon the particle is defined through the function $f(\gamma, t)$. A generalized billiard can be both considered in the framework of Newtonian mechanics and of the relativity theory, see Refs. 4–9.

The generalized reflection law is natural: it both reflects the fact that the walls of the container are at rest, and that the action of the boundary on the particle is an elastic push. A generalized billiard is an approximation of the model with real moving boundaries if the initial velocities of the particles are sufficiently large. From the physical point of view, generalized billiards describe gas consisting of finitely many particles in a container, where the container walls either heat up or cool down (a nonequilibrium gas).

A Newtonian generalized billiard in a parallelepiped is a construction that goes back to Poincaré[1] (Poincaré's original model is a system of finitely many particles, that move in a parallelepiped under the influence of external forces, caused by an external hot body). Relativistic billiards (with moving walls) were considered by Fermi,[10] as a model of particles moving between cosmic objects in the case when the particles interact with the objects' (magnetic) fields. Fermi also considered an "averaged" model, where the walls did not move, but the particle acquired some additional energy at every collision. This model is a generalized relativistic billiard with the "monotone" action of the boundary, i.e., when $\frac{\partial f}{\partial t}(\gamma, t^*) \geq c > 0$. We refer to Refs. 5, 9, where the exponential growth of the particle energy (the result that originally belongs to Fermi) and the existence of attractors in the velocity phase space were rigorously proved, also in the presence of external fields.

Generalized billiards were originally introduced and studied because of their importance for foundations of thermodynamics and nonequilibrium statistical mechanics (Loschmidt reversibility paradox and the justification of the second law of thermodynamics). A key step in handling these problems is the transition from the Newtonian to relativistic billiards. For classical billiards, when $\frac{\partial f}{\partial t}(\gamma, t^*) = 0$, there is no difference between these two cases: it is the same dynamical system. However, in the general case, when $\frac{\partial f}{\partial t}(\gamma, t^*) \neq 0$, these two systems become different. A generalized billiard in the Newtonian case may have an invariant measure (equivalent to the phase volume), and thus be a conservative system. Such an invariant measure has been previously found for a parallelepiped.[4] A generalized relativistic billiard is always dissipative. Thus, the Gibbs entropy is constant in the Newtonian case, whereas it may increase in the relativistic case.

The proofs of the entropy growth for generalized billiards in a parallelepiped with the periodic action of the boundary were given in Ref. 4, and the case of an arbitrary domain with the "monotone" action of the boundary was considered in Ref. 5. In this article we consider generalized billiards in a ball: in the Newtonian case, this is the main example, when a generalized billiard with a smooth boundary possesses a "universal" invariant measure, i.e., the measure with time-independent density, that is invariant for all actions of the boundary (i.e., for all functions $f(\gamma, t)$). From the physical point of view, it is natural to consider the Gibbs entropy defined with respect to a universal measure: the definition of the entropy cannot depend on the way the vessel walls heat up or cool down.

First we consider Newtonian generalized billiards. We prove a theorem on necessary and sufficient conditions for a generalized billiard to possess a smooth universal invariant measure: the corresponding classical billiard should have a first integral of a special type. Typically, generalized billiards do not have a universal invariant measure, and in particular, generalized Sinai billiards do not possess such an invariant measure. In the two-dimensional case, due to the Birkhoff conjecture, elliptic billiards are the only generalized billiards with smooth boundary, that possess a universal measure. We show that a universal invariant measure exists for a Newtonian generalized billiard in a ball, thus its Gibbs entropy, defined with respect to this measure, is constant.

Then we consider the relativistic case. We find the conditions when the Gibbs entropy of the generalized relativistic billiard in a ball, taken with respect to the same measure, as in the Newtonian case, grows. If the action of the boundary is periodic (a "pulsating" ball), then the Gibbs entropy grows, if a certain integral condition, equivalent to one at Ref. 4 is satisfied, and if the initial probability density is nonzero only near the centre of the ball, and the initial particles' velocities are close enough to the velocity of light. The integral condition determines the time direction, in which the entropy increases. The physical meaning of this condition is that the walls are hotter than the gas. In our models it plays the same role as the Boltzmann collision integral, cf. Ref. 11.

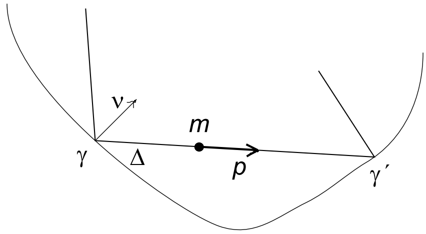

Fig. 1. Coordinate system for a billiard flow.

2. NEWTONIAN GENERALIZED BILLIARDS

Consider a dynamical system on $M = \{x_1, \ldots, x_n\}$, that is given by an invertible mapping $T : M \to M$. Suppose that this system has an invariant measure μ with a density $\rho(x)$, i.e.,

$$d\mu = \rho(x)d^n x, \quad x = (x_1, \ldots, x_n).$$

Let $T : x^0 \to x^1$. Then the Jacobian of this mapping equals

$$J = \frac{\rho(x^0)}{\rho(x^1)}.$$

The converse is also true: if for any point x^0, $x^1 = Tx^0$, the Jacobian of an invertible mapping T equals

$$J = \frac{\rho(x^0)}{\rho(x^1)}$$

for some function ρ, then this function is the density of the invariant measure.

Consider now a Newtonian generalized billiard in a 3-dimensional domain Π with a boundary Γ: a particle m of mass 1 moves in the interior of the domain Π along the straight lines, and reflects from the boundary according to the generalized billiard law, that we describe in details below. We will here consider a billiard *flow* (i.e., a continuous system with the six-dimensional phase space).

First, we introduce the coordinates in the particle phase space. As the particle mass is 1, and the metric in the domain Π is Euclidean, we can identify the velocities and the momenta of the particle. The particle position is uniquely determined by the particle momentum p, the point $\gamma \in \Gamma$ of the latest collision of the particle with the boundary, and the distance $\Delta = \text{dist}(\gamma, m)$ between the particle m and the point γ, see Fig. 1. We will further consider the projection p^ν of the particle momentum p to the normal ν to the boundary Γ at the point γ, and the projection p^τ of the momentum to the tangent plane to Γ at γ. As the particle coordinates, we now take p^ν, p^τ, γ and Δ.

Remark. The 2-vector p^τ is defined uniquely. However, to define the components of this vector, one has to fix a coordinate system in the tangent plane $T_\gamma\Gamma$ first. There is an ambiguity in the choice of basic vectors on the tangent plane $T_\gamma\Gamma$ (moreover, they cannot be chosen globally, unless Γ is diffeomorphic to a torus). However the subsequent results will not depend on the particular choice of these coordinates. Notice that in the two-dimensional case, p^τ is one-dimensional and is well-defined.

We now define the Newtonian generalized reflection law. Suppose that the particle collides with the boundary Γ at a point γ at time t. Then at the collision, the normal component of the particle momentum transforms like $-p^\nu \to p^\nu + P$, where $P = P(t, \gamma)$ is a given function of time and of a point at the boundary. The tangential component of the momentum remains the same. This means that at the impact, the particle momentum transformation is such that as if the (infinitely-heavy) boundary moved along its normal with the velocity $P/2$, see Introduction. From the physical point of view, generalized billiards describe gas consisting of finitely many particles in a vessel, where the vessel walls either heat up or cool down (a nonequilibrium gas).

Let now a particle with coordinates p_0^ν, p_0^τ, γ_0, Δ_0 be strictly inside of Π at time t_0, and suppose that on a time interval $[t_0, t_1]$ the particle collides with the boundary Γ only once, and $\Delta(t_1) \neq 0$. Then for a sufficiently small neighbourhood of the point p_0^ν, p_0^τ, γ_0, Δ_0, all the trajectories that start in this neighbourhood at time t_0, intersect the boundary Γ on the time interval $[t_0, t_1]$ only once. We fix the time t_0, and in this neighbourhood we consider a mapping

$$T : (p^\nu, p^\tau, \gamma, \Delta) \to (p^{\nu\prime}, p^{\tau\prime}, \gamma', \Delta')$$

which is a shift along the phase flow by time $t_1 - t_0$.

Proposition 2.1. *The Jacobian for the mapping T equals*

$$J = \frac{p^\nu}{p^{\nu\prime} - P(t_{\text{hit}}, \gamma')} \frac{||v'||}{||v||}, \tag{1}$$

where $t_{\text{hit}} = t_0 + (\text{dist}(\gamma', \gamma) - \Delta)/||v||$ *is the collision time of the particle with the boundary and v is the particle velocity.*

We note again that, as the particle mass is 1, in the Newtonian case we can identify the particle velocity with the momentum, i.e., $p = v$. In the relativistic case this, of cause, is not true.

Proof. The mapping T can be written as a composition of the following mappings:

$$T = T_4 \cdot T_3 \cdot T_2 \cdot T_1,$$

where the mapping

$$T_1 : (\gamma, \Delta, p^\tau, p^\nu) \to (\gamma, t_{\text{hit}}, p^\tau, p^\nu),$$

where $t_{\text{hit}} = t_0 + (\text{dist}(\gamma', \gamma) - \Delta)/||v||$, determines the time of the collision of the particle with the boundary (the distance dist(γ, γ') is expressed as a function of γ and p), the mapping

$$T_2 : (\gamma, t_{\text{hit}}, p^\nu, p^\tau) \to (\gamma', t_{\text{hit}}, \tilde{p}_\nu, p^{\tau\prime}),$$

is a bouncing map for a classical billiard; the mapping

$$T_3 : (\gamma', t_{\text{hit}}, \tilde{p}_\nu, p^{\tau\prime}) \to (\gamma', t_{\text{hit}}, p'_\nu, p^{\tau\prime}),$$

where $p'_\nu = \tilde{p}^\nu + P(t_{\text{hit}}, \gamma')$ is the generalized billiard reflection law, and, at last, the mapping

$$T_4 : (\gamma', t_{\text{hit}}, p'_\nu, p^{\tau\prime}) \to (\gamma', \Delta', p'_\nu, p^{\tau\prime}),$$

where $\Delta' = (t_1 - t_{\text{hit}})||v'||$ defines the new phase space coordinate Δ'.

The Jacobian for the mapping T_1 equals $J_1 = -1/||v||$, and the Jacobian for the mapping T_4 equals $J_4 = -||v'||$. The mapping T_2 is the bouncing map for a classical billiard, its Jacobian equals

$$J_2 = p^\nu / \tilde{p}^\nu,$$

see, e.g., Ref. 12. At last, the Jacobian for the mapping T_3 equals 1. Thus, the Jacobian for the mapping T is

$$J = \frac{p^\nu}{\tilde{p}^\nu} \frac{||v'||}{||v||}, \tag{2}$$

where $\tilde{p}^\nu = p^{\nu\prime} - P(t_{\text{hit}}, \gamma')$. □

Remark. We have formulated Proposition 2.1 for the 3D case. However, it is also true in all dimensions.

3. INVARIANT MEASURE FOR A NEWTONIAN GENERALIZED BILLIARD

We will call a stationary measure (i.e., the density of the measure does not depend explicitly on time, and neither on boundary action) on the billiard phase space "universal," if it is invariant for any boundary action $P(t, \gamma)$.

We show here that in the general case one cannot expect that a generalized billiard flow in a given domain has a universal invariant measure.

Theorem 3.1. *A universal smooth invariant measure for a generalized billiard flow exists for all boundary actions $P(t, \gamma)$, if and only if the corresponding classical billiard has a first integral of the form $p^{\nu}\ F(p^{\tau}, \gamma)$.*

From this theorem follows immediately that an invariant measure exists for generalized billiards in a ball and in a parallelepiped. For the ball, a first integral is p^{ν}. For a parallelepiped, given in the Cartesian coordinates x_1, x_2, x_3 by $|x_i| \leq$ const, a first integral is the product of projections of the momentum to the coordinate axes x_1, x_2, x_3, which can always be written as $p^{\nu} p_1^{\tau} p_2^{\tau}$ (ν, τ_1 and τ_2 are coordinate axes obtained from x_1, x_2, x_3 by permutation of indices), cf. Ref. 4.

If a classical billiard is nonintegrable, then the corresponding generalized billiard has no universal invariant measure, as the kinetic energy integral cannot be put into the above form. Notice that a billiard flow has the same integrals as the billiard bouncing mapping (first integrals for billiard flows, expressed in coordinates γ, Δ, p, cannot depend on Δ).

Proof. For any point in the interior of Π, consider a shift along the billiard phase trajectory $T : (p^{\nu}, p^{\tau}, \gamma, \Delta) \to (p^{\nu\prime}, p^{\tau\prime}, \gamma', \Delta')$, defined in Sec 2. One can show that it is enough to prove the theorem for the mapping T only.

Assume that a universal invariant measure exists, ρ being its density. In this case, the Jacobian for the mapping T should equal

$$J = \frac{\rho(x)}{\rho(x')}, \quad x = (\gamma, \Delta, p^{\nu}, p^{\tau})$$

(the expression of x' as a function of x contains function P, its arguments are determined in Proposition 2.1).

We first notice, that, as the measure is universal, the density ρ is also a density of invariant measure for the corresponding classical billiard, i.e., when function $P \equiv 0$. Let

$$\rho = \frac{p^{\nu}}{||v|| f(p^{\nu}, p^{\tau}, \gamma, \Delta)}.$$

As ρ is the density of an invariant measure for the classical billiard, and $||v|| = ||v'||$ for $P \equiv 0$, the equality

$$\frac{p^{\nu}}{p^{\nu\prime}} \equiv \frac{\rho(x)}{\rho(x')} = \frac{p^{\nu}}{p^{\nu\prime}} \frac{f(p^{\nu\prime}, p^{\tau\prime}, \gamma', \Delta')}{f(p^{\nu}, p^{\tau}, \gamma, \Delta)}$$

should hold for any two points $x, x' = Tx$ on the trajectory of the classical billiard (cf. (1)). Thus,

$$f(p^{\nu\prime}, p^{\tau\prime}, \gamma', \Delta') = f(p^{\nu}, p^{\tau}, \gamma, \Delta),$$

which means that function f is a first integral for a classical billiard (and it does not depend on Δ).

The condition for ρ to be the density of the invariant measure for the generalized billiard is:

$$\frac{p^{\nu}}{\tilde{p}^{\nu}} \frac{||v'||}{||v||} \equiv \frac{p^{\nu}}{p^{\nu\prime}} \frac{||v'||}{||v||} \frac{f(p^{\nu\prime}, p^{\tau\prime}, \gamma')}{f(p^{\nu}, p^{\tau}, \gamma)} \tag{3}$$

for any function P, where $p^{\nu\prime} = \tilde{p}^{\nu} + P$, see (2). Take now $P = \text{const}$. We substitute the equality $p^{\nu\prime} = \tilde{p}^{\nu} + P$ into identity (3) and differentiate it by P at $P = 0$:

$$0 = \frac{p^{\nu}}{\tilde{p}^{\nu}} \frac{\partial f(\tilde{p}^{\nu}, p^{\tau\prime}, \gamma')/\partial \tilde{p}^{\nu}}{f(p^{\nu}, p^{\tau}, \gamma)} - \frac{p^{\nu}}{(\tilde{p}^{\nu})^2} \frac{f(\tilde{p}^{\nu}, p^{\tau\prime}, \gamma')}{f(p^{\nu}, p^{\tau}, \gamma)},$$

thus

$$\frac{\partial f(\tilde{p}^{\nu}, p^{\tau\prime}, \gamma')}{\partial \tilde{p}^{\nu}} = \frac{f(\tilde{p}^{\nu}, p^{\tau\prime}, \gamma')}{\tilde{p}^{\nu}}.$$

This is a differential equation for a function $f = f(\tilde{p}^{\nu})$, where $p^{\tau\prime}$ and γ' are parameters. Solving it, we get $f(p^{\nu}, p^{\tau}, \gamma) = p^{\nu} F(p^{\tau}, \gamma)$. Thus, if the measure exists, then a first integral should necessarily have the form $f = p^{\nu} F(p^{\tau}, \gamma)$.

One can check by direct computations that the converse is true—by substituting this integral into relation (3). □

Corollary 3.2. *For two-dimensional generalized Newtonian billiards a universal invariant measure exists for billiards in ellipses.*

Proof. Theorem 3.1 is also true in the two-dimensional case, cf. Remark in Sec 2. For the classical billiards in ellipses, there is a first integral

$$\frac{\cos^2\theta - \epsilon^2 \cos^2\phi}{1 - \epsilon^2 \cos^2\phi},$$

where θ is the angle between the momentum and the tangent line to the ellipse and ϕ is the coordinate on the ellipse, which is the angle made by the same tangent line with the fixed vertical axis, and ϵ is the eccentricity of the ellipse. Thus, $\sin\theta$ is the projection of the unit momentum to the normal vector. Rearranging this expression, we ge

$$\frac{-\sin^2\theta + (1-\epsilon^2\cos^2\phi)}{1-\epsilon^2\cos^2\phi} = -\frac{\sin^2\theta}{1-\epsilon^2\cos^2\phi} + 1.$$

Thus, the function

$$\frac{\sin^2\theta}{1-\epsilon^2\cos^2\phi}$$

is a first integral, and we can take its square root, as $\sin\theta \geq 0$ for all $\theta \in [0, \pi]$, to get the form of the first integral, required by Theorem 3.1. □

The famous Birkhoff conjecture states that in 2D, the only integrable billiard with a smooth convex boundary is a billiard in an ellipse (see Ref. 3). From this conjecture and Corollary 3.2 follows, that in the 2-dimensional case, *among generalized billiards in compact smooth convex domains, billiards in ellipses is the only case, when the universal invariant measure exists*.

4. INVARIANT MEASURE AND GIBBS ENTROPY FOR NEWTONIAN GENERALIZED BILLIARD IN A BALL

Let now the domain Π be a ball. A classical billiard in a ball has a first integral p^ν, thus, by Theorem 3.1, the "universal" invariant measure for a generalized Newtonian billiard in a ball exists and has the density

$$\rho = \frac{1}{||v||} \tag{4}$$

in the coordinates defined in Sec 2. We assume that the particle velocity tangential component v^τ is not zero (otherwise the system becomes essentially one-dimensional; this case was considered in details in Ref. 4). Then, the measure with the density (4) is well-defined: for a given trajectory, the particle velocity $||v|| \geq \delta > 0$, where the constant $\delta = ||v^\tau||$, as at every impact the tangential component v^τ of the particle velocity is preserved.

We now remind the definition of the Gibbs entropy. Let a dynamical system be given on a phase space K:

$$\dot{x} = X(x), \tag{5}$$

where $x = (x_1, \ldots, x_n)$ are local coordinates on K, and let μ be some measure on K. Following Gibbs,[13] we introduce a probability measure on K, which at the initial instant $t = 0$ has density $\psi(x) \geq 0$:

$$\int_K \psi \, d\mu = 1.$$

We assume that this measure is transferred by the phase flow g^t of System (5) (and thus its density $\psi_t(x)$ depends on time t):

$$\int_{g^t V} \psi_t d\mu = \int_V \psi d\mu \tag{6}$$

for any domain $V \subset K$.

Remark. While the probability measure is transferred by the phase flow g^t, the density function $\psi_t(x)$ may not be a first integral of System (5). Let, for example, $x_1, \ldots, x_n$ be the Cartesian coordinates, and $d\mu = dx_1 \wedge \ldots \wedge dx_n$. Then relation (6) is equivalent to the Liouville equation

$$\frac{\partial \psi_t}{\partial t} + \mathrm{div}(\psi_t X) = 0 = \frac{\partial \psi_t}{\partial t} + \sum_{i=1}^{n} \frac{\partial \psi_t}{\partial x_i} X_i + \psi_t \mathrm{div}\, X,$$

and one can see that $\psi_t \neq 0$ is a (time-dependent) first integral of System (5) if and only if $\mathrm{div}\, X(x) = 0$.

The Gibbs entropy of System (5) with respect to the measure μ is by definition given by

$$H(t) = -\int_K \psi_t \ln \psi_t d\mu. \tag{7}$$

The statement below belongs to Poincaré:[(1)]

Theorem 4.1. *Let the measure μ be invariant for dynamical system* (5). *Then the Gibbs entropy $H(t)$ is constant.*

Notice that if the measure μ, is not invariant, then one cannot guarantee the conservation of the entropy.

The proof of the theorem is straight-forward: under the conditions of the theorem, ψ_t is a first integral of System (5) (cf. the Remark above), and it can be written as $\psi_t(x) = \psi(g^{-t}x)$. Thus, as the measure μ is invariant, the Gibbs entropy (7) is constant. We refer to Ref. 4 for details and application of the Poincaré theorem to generalized billiards.

To apply the Poincaré theorem to a dynamical system, we only have to check that this system possesses an invariant measure. For the Newtonian generalized billiard in a ball, such invariant measure exists, and is given by the density (4). Thus the Gibbs entropy $H(t)$, defined with respect to this measure, is constant. In the next section we consider the Gibbs entropy of the relativistic generalized billiard in a ball, with respect to the same measure. We show that under some natural conditions the Gibbs entropy grows.

5. RELATIVISTIC GENERALIZED BILLIARDS IN A BALL

We now consider a relativistic generalized billiard in the 3-dimensional (space-) domain Π with the boundary Γ. From the physical point of view, it is natural to consider a nonequilibrium gas, which consists of particles that move fast enough, in the framework of the relativity theory. We measure the velocities proportional to the velocity of light c, i.e., here we take $c = 1$. We refer to Ref. 10 (and to Fefs. 4 and 8 for details and proofs) for definitions and expressions for the relativistic billiard reflection laws. Here we give expressions for the momentum and velocity transformations. The particle velocity v and momentum p are related as

$$p = \frac{1}{\sqrt{1 - ||v||^2}} v,$$

Here $|| \cdot ||$ is the usual Euclidean norm.

Let the particle fall to the infinitely-heavy horizontal wall, which in turn moves in the vertical direction with the velocity V. After the impact the projection of the momentum to the tangent plane to the wall remains the same: $p^{\tau\prime} = p^{\tau}$, while the projection to the normal to the wall $p^{\nu\prime}$ after the impact equals

$$p^{\nu\prime} = (-p^{\nu})\frac{1+V}{1-V} + \frac{2V}{1-V^2}(\sqrt{||p||^2 + 1} - p^{\nu}) \tag{8}$$

($||p||$ is the 3-dimensional Euclidean length of the momentum vector). We have assumed that $p^{\nu} < 0$, which means that the particle falls to the wall.

The velocity transformation at the collision with a moving wall is given by the following expression, see, e.g., Ref. 9. Let the particle hit the infinitely-heavy wall with the velocity v, with the normal component $v_{\nu} < 0$, and, as above, the wall moves along its normal with the velocity V. Then after the impact the particle velocity v' equals

$$v'_{\nu} = -\frac{v_{\nu} - 2V + V^2 v_{\nu}}{1 - 2V v_{\nu} + V^2}, \quad v'_{\tau} = \frac{v_{\tau}(1 - V^2)}{1 - 2V v_{\nu} + V^2}, \tag{9}$$

v_{τ} is the tangential component of the velocity. It is interesting to note, that, while the tangential component of the momentum is preserved, the tangential component of the velocity changes at the impact with the moving wall.

We remind again, that the essence of the generalization is that at the collision of the particle with the boundary Γ, the particle momentum and the velocity are transformed as above, as if the particle undergoes an elastic push by an infinitely-heavy wall, which moves with the velocity V, while the boundary Γ itself does not move. The function $V(t, \gamma)$ will be refereed to as the boundary action velocity. We notice that $|V| < 1$, i.e., the boundary action velocity cannot be equal to the speed of light.

Now, let the domain Π be a ball of diameter l. For a classical billiard, both the normal and the tangential components of the momentum are preserved along the billiard trajectory. We assume that the boundary action velocity V is a function of time t only.

Let Δ, γ, p^{ν} and p^{τ} be the coordinates introduced in Sec 2. Take a point Δ_0, γ_0, p_0^{ν} and p_0^{τ} strictly inside the ball Π, and fix a moment of time t_0. We consider the same mapping T as in Sec 2: suppose that on a time interval $[t_0, t_1]$ the particle collides with the boundary only once, and $\Delta(t_1) \neq 0$. Then the same is true for all trajectories which start in a small neighbourhood of the initial point $\Delta_0, \gamma_0, p_0^{\nu}, p_0^{\tau}$, and in this neighbourhood we define the mapping

$$T : (\gamma, \Delta, p^{\nu}, p^{\tau}) \to (\gamma', \Delta', p^{\nu\prime}, p^{\tau\prime})$$

which is a shift by time $t_1 - t_0$ along the phase trajectory.

Proposition 5.1. *Let p^{τ} be bounded. Then the Jacobian J for the mapping T at $p^{\nu} \to \infty$ tends to*

$$J \to J_{\infty} = \left(1 + V\left(t_{\text{hit}}^{\infty}\right)\right) / \left(1 - V\left(t_{\text{hit}}^{\infty}\right)\right),$$

where $t_{\text{hit}}^{\infty} = t_0 + (l - \Delta)$.

We represent the mapping T as $T = T_4 \cdot T_3 \cdot T_2 \cdot T_1$, where

$$T_1 : (\gamma, \Delta, p^{\nu}, p^{\tau}) \to (\gamma, t_{\text{hit}}, p^{\nu}, p^{\tau}),$$

with $t_{\text{hit}} = t_0 + (\text{dist}(\gamma, \gamma') - \Delta)/||v||$, determines the impact moment, the mapping

$$T_2 : (\gamma, t_{\text{hit}}, p^{\nu}, p^{\tau}) \to (\gamma', t_{\text{hit}}, p^{\nu}, p^{\tau}), \quad \gamma' = \gamma + G(v)$$

defines the new value of the boundary coordinate γ,

$$T_3 : (\gamma', t_{\text{hit}}, p^{\nu}, p^{\tau}) \to (\gamma', t_{\text{hit}}, p^{\nu\prime}, p^{\tau\prime}),$$

where $p^{\nu\prime}$ is defined by relation (8) and $\rho^{\tau\prime} = \rho^{\tau}$, is the relativistic reflection law, and the mapping

$$T_4 : (\gamma', t_{\text{hit}}, p^{\nu\prime}, p^{\tau\prime}) \to (\gamma', \Delta', p^{\nu\prime}, p^{\tau\prime}), \quad \Delta' = (t_1 - t_{\text{hit}})||v'||$$

determines the spacial coordinate Δ'. The relation $\gamma' = \gamma + G(v)$ makes sense, since any billiard trajectory in a ball always belongs to some plane, which passes through the centre of the ball. This place crosses the boundary sphere by the circle, and γ can be expressed through the angle coordinate on this circle.

The Jacobians for the mappings T_1 and T_4 are $-1/||v||$ and $-||v'||$ correspondingly. The Jacobian for the mapping T_2 equals 1, while the Jacobian for the mapping T_3 is

$$J_3 = \frac{1 + V}{1 - V} + O\left(\frac{1 + ||p^{\tau}||^2}{(p^{\nu})^2}\right).$$

Thus, when $p^\nu \to \infty$, the Jacobian for the mapping T tends to $(1 + V(t_{\rm hit}^\infty))/(1 - V(t_{\rm hit}^\infty))$. The limit value of the impact moment $t_{\rm hit}^\infty$ equals $t_0 + (l - \Delta)$, as, when $p^\nu \to \infty$ and p^τ being fixed, the velocities tend to $v^\tau \to 0$, $v^\nu \to 1$. □

6. GIBBS ENTROPY FOR GENERALIZED RELATIVISTIC BILLIARDS IN A BALL

The Jacobian J of the mapping T plays the key role in the proof of the Gibbs entropy growth, see Ref. 4. Take the measure μ with the density ρ, defined by relation (4). From (4) and (6) we get, that under the mapping T,

$$\psi' = \frac{1}{J}\frac{||v'||}{||v||}\psi. \tag{10}$$

Thus,

$$\begin{aligned} H' - H &= \int_K \psi \ln \psi d\mu - \int_K \psi' \ln \psi' d\mu' = \int_K (\ln \psi - \ln \psi')\psi d\mu \\ &= \int_K (\ln ||v|| - \ln ||\hat{v}|| + \ln J)\psi d\mu \end{aligned} \tag{11}$$

Suppose that the limit Jacobian $J_\infty > 1$. Then, obviously, the Gibbs entropy grows, if the normal component of the velocity v^ν is close enough to the velocity of light 1.

Let ξ_1, ξ_2, ξ_3 be the Cartesian space-coordinates, such that the ball is given by

$$\xi_1^2 + \xi_2^2 + \xi_3^2 \le \frac{l^2}{4}.$$

Consider first the "monotone" action of the boundary: $V \ge V_0 > 0$.

Theorem 6.1. *For the generalized relativistic billiard in a ball with the monotone action of the boundary, the Gibbs entropy, defined with respect to the measure with the density (4), grows faster than some linear function of time, provided at $t = 0$, the probability density ψ is positive only in a sufficiently small neighbourhood of the subset $||v|| = 1$, $\xi = 0$.*

Proof. Under conditions of the theorem, $v^\tau \to 0$, $||v|| \to 1$ as $t \to \infty$ (see Ref. 9), thus $||p^\tau||/p^\nu \to 0$ and $p^\nu \to \infty$. From Proposition 5.1 follows that if the normal component of the velocity v^ν is close enough to 1, then the Jacobian $J \ge \delta > 1$. The theorem follows now from relation (11). □

Suppose now that the boundary action velocity may change the sign, and is periodic: $V(t+1) = V(t)$. To ensure the Gibbs entropy growth, one has to claim that a certain integral condition must be fulfilled.

Theorem 6.2. *Let the integral*

$$I = \int_0^1 \ln \frac{1+V(t)}{1-V(t)} dt \geq \delta > 0 \tag{12}$$

Then there exists a constant $N \in \mathbb{N}$ such that if the ball diameter $l \neq \frac{p}{q}$, where $p, q \in \mathbb{N}$, p/q is an irreducible fraction, such that $q < N$, then there exist constants $C_1 > 0$, C_2, such that the Gibbs entropy H, defined with respect to the measure with the density (4), satisfies the following estimate:

$$H(t) \geq H(0) + C_1 t + C_2$$

for all $t \geq 0$, provided that at $t = 0$, the probability density ψ is positive only in a sufficiently small neighbourhood of the subset $||v|| = 1$, $||\xi|| = 0$.

Remarks. 1. Formally, Theorem 6.1 is not a corollary of Theorem 6.2, as for the monotone action of the boundary we do not need any condition on the ball diameter l. Notice that, unlike Ref. 4, here the integral condition (12) itself does not depend explicitly on the ball diameter l. The physical meaning of the integral condition (12) is that the walls are hotter than the gas.
2. Theorem 6.2 is especially interesting (and physically important), when

$$\int_0^1 V(t)dt = 0,$$

i.e., the wall's "motion" is periodic (a "pulsating" ball). For almost all choices of such function $V(t)$, the integral $I \neq 0$. As an example of such function one can take

$$V(t) = \epsilon(Q_1 \cos 2\pi kt + Q_2 \cos 4\pi kt),$$

see Ref. 4, where it was proved than Inequality (12) is satisfied for such boundary action velocity. Here $\epsilon > 0$ is a small parameter, k is an integer, and $Q_1 \neq 0$, Q_2 are constants, such that $Q_2 k > 0$.

Proof. The proof is similar to the proofs in Refs. 4 , 9. Let us denote

$$A(t) = \frac{1+V(t)}{1-V(t)}.$$

If l is an irrational number, then the rotation of the circle $t \to t + l \pmod 1$ is a uniquely ergodic mapping, and by the ergodic theorem, the sum

$$\frac{1}{n}\sum_{k=0}^{n-1} \ln A(t + kl)$$

converges uniformly to the integral I.

Suppose now that $\frac{p}{q} \in L$ is an irreducible fraction. Then the sum

$$\frac{1}{q}\sum_{k=0}^{q-1} \ln A\left(t + \frac{kp}{q}\right). \tag{13}$$

is exactly an integral sum of the integral I, as all the q points $t + pk/q \pmod 1$ are different on the circle S^1 $(k = 0 \ldots q - 1)$. If the value of q is large enough $(q \geq N \gg 1)$, then the sum (13) approximates the integral I with a given precision for all such $p/q \in L$, as L is compact. Notice, that if we fix the value p/q, then obviously the sum

$$\frac{1}{q}\sum_{k=0}^{q-1} \ln A\left(t + \frac{kp}{q} + g_k\right), \tag{14}$$

where g_k are arbitrary values, that satisfy inequalities $|g_k| \leq \epsilon$, approximates the integral I with the same precision, provided ϵ is small enough.

Thus, for a given diameter l, that satisfies the conditions of the theorem, and for $t \pmod 1$, there are constants $\tilde{C}_1 > 0$, $\tilde{C}_2$, such that for any $n \in \mathbb{N}$

$$\sum_{k=0}^{n} \ln A(t + kl) > \tilde{C}_1 n + \tilde{C}_2. \tag{15}$$

The transformation of the velocities can be separated from the transformation of the configuration space variables $\gamma \in \Gamma$, as the values $||v^\tau, |v^\nu|$ *after* a collision and *before* the next collision are exactly the same: the boundary Γ of our domain is a sphere. Let $|v^\nu| = 1 - w^2$. Obviously, $w = 0$, $v^\tau = 0$ is an invariant manifold for the generalized billiard (notice explicit time dependence).

Let t be a moment of time, when the particle hits the boundary, and let w, v^τ be the particle velocity *before* this collision. Consider a mapping T, that sends a point w, v^τ, t to $\hat{w}$, $\hat{v}^\tau$, $\hat{t}$, where $\hat{t}$ is the moment of the next collision, and $\hat{w}$, $\hat{v}^\tau$ is the velocity before the next collision.

Using (9), one can immediately see that in the linear approximation,

$$\hat{w} = \frac{1 - V(t)}{1 + V(t)} w, \quad \hat{v}^\tau = \frac{1 - V(t)}{1 + V(t)} v^\tau, \quad \hat{t} = t + l.$$

Consider the mapping T^n for a big value of n. Due to (15), the product

$$\prod_{k=0}^{n-1} \frac{1 - V(t + kl)}{1 + V(t + kl)} = \prod_{k=0}^{n-1} \frac{1}{A(t + kl)} < 1.$$

Thus, the mapping T^n is contracting in the velocities in the linear approximation, and one can show that its invariant manifold $w = 0$, $v^\tau = 0$ is indeed asymptotically Lyapunov-stable (cf. Ref. 9): the key point is that if the values of the velocities w and v^τ are small enough, then this cannot influence convergence due to (14).

The theorem now follows, as when the velocity $|v^\nu| \to 1$, the momentum components $p^\nu \to \infty$ and $||p^\tau||/p^\nu \to 0$ as $t \to \infty$. □

ACKNOWLEDGMENTS

We wish to thank the reviewers for useful comments and corrections of the manuscript. L.D. Pustyl'nikov was supported by the Russian Foundation for Basic Research, grant no. 02-01-01067.

REFERENCES

1. H. Poincaré, Réflexions sur la théorie cinétique des gaz. *J. Phys. Theoret. et Appl.* **5**(4):349–403 (1906).
2. Ya. G. Sinai, Dynamical systems with elastic reflections. Ergodic properties of dispersing billiards. *Russ. Math. Surv.* **25**(2):137–189 (1970).
3. G. Birkhoff. *Dynamical systems*. New York, AMS (1927).
4. L. D. Pustyl'nikov, Poincaré models, rigorous justification of the second law of thermodynamics from mechanics, and the Fermi acceleration mechanism. *Russian Math. Surveys* **50**(1):145–189 (1995).
5. L. D. Pustyl'nikov. The law of entropy increase and generalized billiards. *Russian Math. Surv.* **54**(3):650–651 (1999).
6. L. D. Pustyl'nikov, A new mechanism for particle acceleration and a relativistic analogue of the Fermi–Ulam model. *Theoret. and Math. Phys.* **77**(1):1110–1115 (1988).
7. M. V. Deryabin and L. D. Pustyl'nikov, Generalized relativistic billiards. *Reg. and Chaotic Dyn.* **8**(3):283–296 (2003).
8. M. V. Deryabin and L. D. Pustyl'nikov, On generalized relativistic billiards in external force fields. *Lett. Math. Phys.* **63**(3):195–207 (2003).
9. M. V. Deryabin and L. D. Pustyl'nikov, Exponential attractors in generalized relativistic billiards. *Comm. Math. Phys.* **248**(3):527–552 (2004).
10. E. Fermi, On the origin of the cosmic radiation. *Phys. Rev.* **75**:1169 (1949).
11. G. E. Uhlenbeck and G. W. Ford. *Lectures in Statistical Mechanics*. AMS, Providence, RI (1963).
12. V. V. Kozlov and D. V. Billiards, Treshchev. *A Genetic Introduction to the Dynamics of Systems with Impacts*. Translations of Mathematical Monographs, 89. AMS, Providence, RI, 1991. viii+171 pp.
13. J. W. Gibbs, Elementary principles in statistical mechanics. In: *The Collected Works of J. W. Gibbs*, Vol. II, Part 1, Yale University Press, New Haven, Conn. (1931).